SECTION 7.7: INTRODUCTION TO HYPERBOLIC FUNCTIONS

In this worksheet, you get to relive all the excitement of those early days of Calculus when you explored properties of the six circular (trigonometric) functions: $\cos(t)$, $\sin(t)$, $\tan(t)$, $\cot(t)$, $\sec(t)$, and $\csc(t)$.

Here, we introduce the **hyperbolic** functions - functions defined in terms of exponential functions that behave hauntingly similarly to the circular functions.

DEFINITIONS:

The **hyperbolic cosine of** t, written $\cosh(t)$ and pronounced 'cosh t' and the **hyperbolic sine of** t, written $\sinh(t)$ and pronounced 'cinch t' are defined by the equations:[1]

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \qquad \sinh(t) = \frac{e^t - e^{-t}}{2}.$$

As with the circular functions, these hyperbolic functions enjoy lots of identities.

NOTE: Like the circular functions, the notation $\cosh^2(t)$ means $(\cosh(t))^2$, $\sinh^2(t)$ means $(\sinh(t))^2$, etc

EXAMPLE 1: Verify the 'Even / Odd' Properties:

1. $\cosh(-t) = \cosh(t)$: $\cosh(-t) = \dfrac{e^{-t} + e^{-(-t)}}{2} = \dfrac{e^{-t} + e^{t}}{2} = \dfrac{e^{t} + e^{-t}}{2} = \cosh(t)\,\checkmark$

2. $\sinh(-t) = -\sinh(t)$: $\sinh(-t) = \dfrac{e^{-t} - e^{-(-t)}}{2} = \dfrac{e^{-t} - e^{t}}{2} = -\dfrac{e^{t} - e^{-t}}{2} = -\sinh(t)\,\checkmark$

EXAMPLE 2: Verify the 'Pythagorean-like' Identity: $\cosh^2(t) - \sinh^2(t) = 1$.

$$\begin{aligned}
\cosh^2(t) - \sinh^2(t) &= \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 \\
&= \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} \\
&= \frac{e^{2t} + 2 + e^{-2t} - e^{2t} + 2 - e^{-2t}}{4} \\
\cosh^2(t) - \sinh^2(t) &= \frac{4}{4} = 1\,\checkmark
\end{aligned}$$

NOTE: This shows coordinates of the form $(\cosh(t), \sinh(t))$ lie on the Unit Hyperbola, $x^2 - y^2 = 1$ in much the same way as coordinates of the form $(\cos(t), \sin(t))$ lie on the Unit Circle, $x^2 + y^2 = 1$. Cool, right?

[1] Hence the 'joke:' Hyperbolic functions are a 'sinh.'

EXAMPLE 3: Verify the following derivative formulas:

1. $D_t\left[\sinh(t)\right] = \cosh(t)$:

$$D_t\left[\sinh(t)\right] = D_t\left[\frac{e^t - e^{-t}}{2}\right] = \frac{1}{2}D_t\left[e^t - e^{-t}\right] = \frac{1}{2}\left(e^t - e^{-t}(-1)\right) = \frac{1}{2}\left(e^t + e^{-t}\right) = \frac{e^t + e^{-t}}{2} = \cosh(t)\,\checkmark$$

2. $D_t\left[\cosh(t)\right] = \sinh(t)$:

$$D_t\left[\cosh(t)\right] = D_t\left[\frac{e^t + e^{-t}}{2}\right] = \frac{1}{2}D_t\left[e^t + e^{-t}\right] = \frac{1}{2}\left(e^t + e^{-t}(-1)\right) = \frac{1}{2}\left(e^t - e^{-t}\right) = \frac{e^t - e^{-t}}{2} = \sinh(t)\,\checkmark$$

EXAMPLE 4: The hyperbolic cosine arises in differential equations as the solution to the 'free hanging cable' problem (whose solution curve is called a 'catenary.')[2]

Verify $y = \dfrac{T}{\rho g}\cosh\left(\dfrac{\rho g x}{T}\right)$ satisfies $y'' = \dfrac{\rho g}{T}\sqrt{1 + (y')^2}$ assuming ρ, g, and T are all positive constants.

HINT: Since $\cosh^2(t) - \sinh^2(t) = 1$, we know $1 + \sinh^2(t) = \cosh^2(t)$. Use this on the right hand side ...

If $y = \dfrac{T}{\rho g}\cosh\left(\dfrac{\rho g x}{T}\right)$, then $y' = \dfrac{T}{\rho g}\sinh\left(\dfrac{\rho g x}{T}\right) D_x\left[\dfrac{\rho g x}{T}\right] = \dfrac{T}{\rho g}\sinh\left(\dfrac{\rho g x}{T}\right)\left(\dfrac{\rho g}{T}\right) = \sinh\left(\dfrac{\rho g x}{T}\right)$.

Hence, $y'' = D_x\left[\sinh\left(\dfrac{\rho g x}{T}\right)\right] = \cosh\left(\dfrac{\rho g x}{T}\right) D_x\left[\dfrac{\rho g x}{T}\right] = \dfrac{\rho g}{T}\cosh\left(\dfrac{\rho g x}{T}\right)$

$$\begin{aligned}
\frac{\rho g}{T}\sqrt{1 + (y')^2} &= \frac{\rho g}{T}\sqrt{1 + \sinh^2\left(\frac{\rho g x}{T}\right)} \\
&= \frac{\rho g}{T}\sqrt{\cosh^2\left(\frac{\rho g x}{T}\right)} \\
\frac{\rho g}{T}\sqrt{1 + (y')^2} &= \frac{\rho g}{T}\cosh\left(\frac{\rho g x}{T}\right) = y''\,\checkmark
\end{aligned}$$

[2]For more on this check out Wikipedia's Catenary Page.

EXAMPLE 5: Graph $f(t) = \cosh(t)$ and $g(t) = \sinh(t)$ on the axes below using Calculus. That is:

1. Use limits as $t \to \pm\infty$ to determine the end behavior of the graphs.

$$\lim_{t\to-\infty} \cosh(t) = \lim_{t\to-\infty} \frac{e^t + e^{-t}}{2} = \infty, \quad \lim_{t\to\infty} \cosh(t) = \lim_{t\to\infty} \frac{e^t + e^{-t}}{2} = \infty$$

$$\lim_{t\to-\infty} \sinh(t) = \lim_{t\to-\infty} \frac{e^t - e^{-t}}{2} = -\infty, \quad \lim_{t\to\infty} \sinh(t) = \lim_{t\to\infty} \frac{e^t - e^{-t}}{2} = \infty$$

2. Use the first derivative to determine intervals of increase and decrease and locate any local extrema.

$f(t) = \cosh(t) \implies f'(t) = \sinh(t)$; $g(t) = \sinh(t) \implies g'(t) = \cosh(t)$. We make Sign Diagrams:

$$\begin{array}{ccccccc} & (-) & 0 & (+) & & \sinh(t) \\ -\infty \longleftarrow & & + & & \longrightarrow & \infty \\ & \uparrow & 0 & \uparrow & & t \\ & -1 & & 1 & & \end{array} \qquad\qquad \begin{array}{ccc} & (+) & \cosh(t) \\ -\infty \longleftarrow & & \longrightarrow \infty \\ & & t \end{array}$$

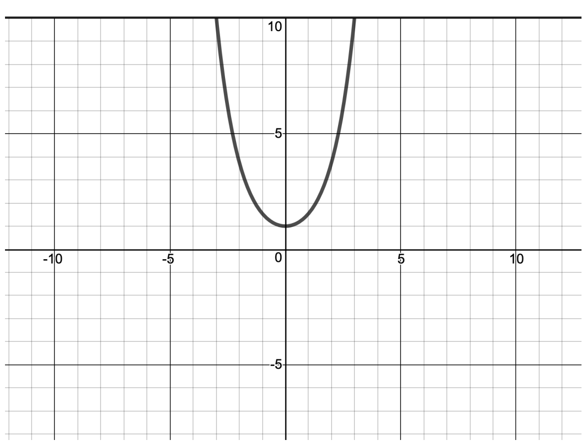
We find $f(t) = \cosh(t)$ is decreasing $(-\infty, 0)$ to a local (absolute!) min at $(0, 1)$ and increasing on $(0, \infty)$. We also find that $g(t) = \sinh(t)$ is always increasing.

3. Use the second derivative to determine the nature of concavity and locate inflection points.

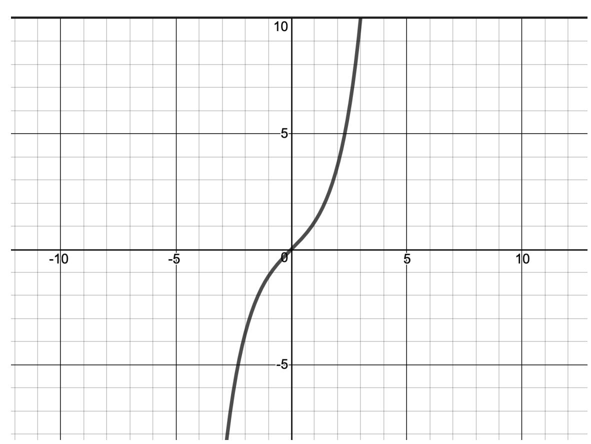
$f(t) = \cosh(t) \implies f''(t) = \cosh(t)$; $g(t) = \sinh(t) \implies g''(t) = \sinh(t)$. We re-use the Sign Diagrams:

$$\begin{array}{ccc} & (+) & \cosh(t) \\ -\infty \longleftarrow & & \longrightarrow \infty \\ & & t \end{array} \qquad\qquad \begin{array}{ccccccc} & (-) & 0 & (+) & & \sinh(t) \\ -\infty \longleftarrow & & + & & \longrightarrow & \infty \\ & \uparrow & 0 & \uparrow & & t \\ & -1 & & 1 & & \end{array}$$

We find $f(t) = \cosh(t)$ is always concave up. We also find that $g(t) = \sinh(t)$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ with an inflection point at $(0, 0)$.



$f(t) = \cosh(t)$



$g(t) = \sinh(t)$

Four additional hyperbolic functions are defined as follows (look familiar?):

DEFINITIONS: (With some reciprocal and quotient identities thrown in for good measure ...)

The **hyperbolic secant of** t, written $\operatorname{sech}(t)$, and the **hyperbolic tangent of** t, written $\tanh(t)$, are defined:

$$\operatorname{sech}(t) = \frac{2}{e^t + e^{-t}} = \frac{1}{\cosh(t)}, \quad \tanh(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}} = \frac{\sinh(t)}{\cosh(t)}.$$

The **hyperbolic cosecant of** t, written $\operatorname{csch}(t)$, and the **hyperbolic cotangent of** t, written $\coth(t)$, are defined:

$$\operatorname{csch}(t) = \frac{2}{e^t - e^{-t}} = \frac{1}{\sinh(t)}, \quad \coth(t) = \frac{e^t + e^{-t}}{e^t - e^{-t}} = \frac{\cosh(t)}{\sinh(t)} = \frac{1}{\tanh(t)}, \quad \text{provided } \sinh(t) \neq 0.$$

EXAMPLE 6: Verify the identity: $1 - \tanh^2(t) = \operatorname{sech}^2(t)$.

We know: $\cosh^2(t) - \sinh^2(t) = 1$, so we divide both sides by $\cosh^2(t)$:

$$\begin{aligned}
\cosh^2(t) - \sinh^2(t) &= 1 \\
\frac{\cosh^2(t) - \sinh^2(t)}{\cosh^2(t)} &= \frac{1}{\cosh^2(t)} \\
1 - \frac{\sinh^2(t)}{\cosh^2(t)} &= \frac{1}{\cosh^2(t)} \\
1 - \tanh^2(t) &= \operatorname{sech}^2(t) \checkmark
\end{aligned}$$

EXAMPLE 7: Verify the identity: $\coth^2(t) - 1 = \operatorname{csch}^2(t)$.

$$\begin{aligned}
\cosh^2(t) - \sinh^2(t) &= 1 \\
\frac{\cosh^2(t) - \sinh^2(t)}{\sinh^2(t)} &= \frac{1}{\sinh^2(t)} \\
\frac{\cosh^2(t)}{\sinh^2(t)} - 1 &= \frac{1}{\sinh^2(t)} \\
\coth^2(t) - 1 &= \operatorname{csch}^2(t) \checkmark
\end{aligned}$$

EXAMPLE 8: Derive a formula for $D_t\left[\operatorname{sech}(t)\right]$ which resembles a derivative formula for $D_t[\sec(t)]$.

$$\begin{aligned}
D_t\left[\operatorname{sech}(t)\right] &= D_t\left[\frac{1}{\cosh(t)}\right] \\
&= D_t\left[(\cosh(t))^{-1}\right] \\
&= (-1)(\cosh(t))^{-2} D_t[\cosh(t)] \\
&= (-1)(\cosh(t))^{-2}\sinh(t) \\
&= -\frac{1}{\cosh(t)}\,\frac{\sinh(t)}{\cosh(t)} \\
D_t\left[\operatorname{sech}(t)\right] &= -\operatorname{sech}(t)\tanh(t)
\end{aligned}$$

This is very reminiscent of the formula $D_t[\sec(t)] = \sec(t)\tan(t)$.

EXAMPLE 9: Derive a formula for $D_t\left[\operatorname{csch}(t)\right]$ which resembles a derivative formula for $D_t[\csc(t)]$.

$$\begin{aligned}
D_t\left[\operatorname{csch}(t)\right] &= D_t\left[\frac{1}{\sinh(t)}\right] \\
&= D_t\left[(\sinh(t))^{-1}\right] \\
&= (-1)(\sinh(t))^{-2} D_t[\sinh(t)] \\
&= (-1)(\sinh(t))^{-2}\cosh(t) \\
&= -\frac{1}{\sinh(t)}\,\frac{\cosh(t)}{\sinh(t)} \\
D_t\left[\operatorname{csch}(t)\right] &= -\operatorname{csch}(t)\coth(t)
\end{aligned}$$

This is very reminiscent of the formula $D_t[\csc(t)] = -\csc(t)\cot(t)$.

EXAMPLE 10: Find a formula for $D_t\left[\tanh(t)\right]$ which resembles a derivative formula for $D_t[\tan(t)]$.

$$
\begin{aligned}
D_t\left[\tanh(t)\right] &= D_t\left[\frac{\sinh(t)}{\cosh(t)}\right] \\
&= \frac{\cosh(t)D_t[\sinh(t)] - \sinh(t)D_t[\cosh(t)]}{(\cosh(t))^2} \\
&= \frac{\cosh(t)\cosh(t) - \sinh(t)\sinh(t)}{\cosh^2(t)} \\
&= \frac{\cosh^2(t) - \sinh^2(t)}{\cosh^2(t)} \\
&= \frac{1}{\cosh^2(t)} \\
D_t\left[\tanh(t)\right] &= \operatorname{sech}^2(t)
\end{aligned}
$$

This is very reminiscent of the formula $D_t[\tan(t)] = \sec^2(t)$.

EXAMPLE 11: Find a formula for $D_t\left[\coth(t)\right]$ which resembles a derivative formula for $D_t[\cot(t)]$.

$$
\begin{aligned}
D_t\left[\coth(t)\right] &= D_t\left[\frac{\cosh(t)}{\sinh(t)}\right] \\
&= \frac{\sinh(t)D_t[\cosh(t)] - \cosh(t)D_t[\sinh(t)]}{(\sinh(t))^2} \\
&= \frac{\sinh(t)\sinh(t) - \cosh(t)\cosh(t)}{\sinh^2(t)} \\
&= \frac{\sinh^2(t) - \cosh^2(t)}{\sinh^2(t)} \\
&= -\frac{\cosh^2(t) - \sinh^2(t)}{\sinh^2(t)} \\
&= -\frac{1}{\sinh^2(t)} \\
D_t\left[\coth(t)\right] &= -\operatorname{csch}^2(t)
\end{aligned}
$$

This is very reminiscent of the formula $D_t[\cot(t)] = -\csc^2(t)$.

EXAMPLE 12: Turn the derivative formulas you've developed into integration formulas:

1. $\int \cosh(x)\,dx = \sinh(x) + C$

2. $\int \sinh(x)\,dx = \cosh(x) + C$

3. $\int \operatorname{sech}(x)\tanh(x)\,dx = -\operatorname{sech}(x) + C$

4. $\int \operatorname{sech}^2(x)\,dx = \tanh(x) + C$

5. $\int \operatorname{csch}(x)\coth(x)\,dx = -\operatorname{csch}(x) + C$

6. $\int \operatorname{csch}^2(x)\,dx = -\coth(x) + C$

EXAMPLE 13: Find the following integrals. Check your answer using differentiation.

1. $\int \sinh(2x)\,dx = \frac{1}{2}\cosh(2x) + C$

2. $\int \operatorname{sech}^2(x)\,\sqrt{\tanh(x)+1}\;dx = \int \operatorname{sech}^2(x)\,(\tanh(x)+1)^{\frac{1}{2}}\;dx$:

   Let $u = \tanh(x) + 1$ so that $du = \operatorname{sech}^2(x)\,dx$ Hence,

   $$\int \operatorname{sech}^2(x)\,(\tanh(x)+1)^{\frac{1}{2}}\;dx = \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(\tanh(x)+1)^{\frac{3}{2}} + C$$

EXAMPLE 14: For $f(t) = \sinh(t)$, $f'(t) = \cosh(t) > 0$ for all t. It follows that $f(t) = \sinh(t)$ is one-to-one.

Find a formula for $f^{-1}(t) = \sinh^{-1}(t)$.

Recall from College Algebra, to find an formula for $f^{-1}(t)$, we set $y = \sinh(t)$ and solve $t = \sinh(y)$.

$$\begin{aligned}
t &= \sinh(y) \\
t &= \frac{e^y - e^{-y}}{2} \\
2t &= e^y - e^{-y} \\
2te^y &= \left(e^y - e^{-y}\right) e^y \\
2te^y &= (e^y)^2 - 1 \\
0 &= (e^y)^2 - (2t)e^y - 1 \\
e^y &= \frac{2t \pm \sqrt{4t^2+4}}{2} && \text{Quadratic Formula} \\
e^y &= \frac{2t \pm 2\sqrt{t^2+1}}{2} \\
e^y &= t \pm \sqrt{t^2+1} \\
e^y &= t + \sqrt{t^2+1} && \text{Since } e^y > 0 \text{, we use the '+' root.} \\
y &= \ln(t + \sqrt{t^2+1}) \\
f^{-1}(t) = \sinh^{-1}(t) &= \ln(t + \sqrt{t^2+1})
\end{aligned}$$

EXAMPLE 15: Find and simplify a formula for $D_t\left[\sinh^{-1}(t)\right] = D_t\left[\ln(t + \sqrt{t^2+1})\right]$.

$$\begin{aligned}
D_t\left[\ln(t + \sqrt{t^2+1})\right] &= \frac{1}{t + \sqrt{t^2+1}} D_t\left[t + \sqrt{t^2+1}\right] \\
&= \frac{1}{t + \sqrt{t^2+1}}\left(1 + \frac{t}{\sqrt{t^2+1}}\right) \\
&= \frac{1}{t + \sqrt{t^2+1}}\left(\frac{\sqrt{t^2+1}}{\sqrt{t^2+1}} + \frac{t}{\sqrt{t^2+1}}\right) \\
&= \frac{1}{t + \sqrt{t^2+1}}\left(\frac{t + \sqrt{t^2+1}}{\sqrt{t^2+1}}\right) \\
D_t\left[\sinh^{-1}(t)\right] = D_t\left[\ln(t + \sqrt{t^2+1})\right] &= \frac{1}{\sqrt{t^2+1}} = \frac{1}{\sqrt{1+t^2}}
\end{aligned}$$

This is reminiscent of $D_t\left[\sin^{-1}(t)\right] = \dfrac{1}{\sqrt{1-t^2}}$.

HOMEWORK: Section 7.7: 23 - 45 odd, 75*